Reachability Problems for Continuous Linear Dynamical Systems

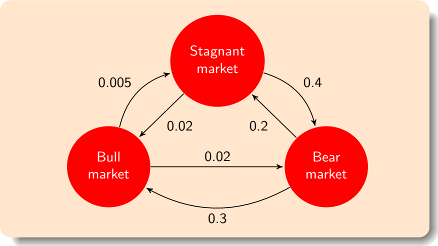
James Worrell

Department of Computer Science, Oxford University

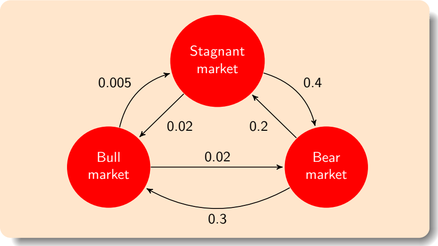
(Joint work with Ventsislav Chonev and Joël Ouaknine)

CONCUR
September 2nd 2015

Reachability for Continuous-Time Markov Chains



Reachability for Continuous-Time Markov Chains

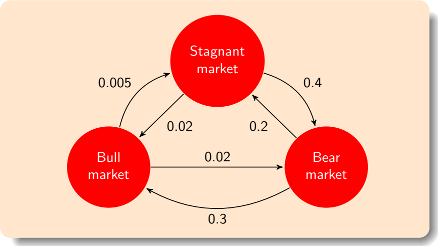


Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

is the **rate matrix**.

Reachability for Continuous-Time Markov Chains



Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

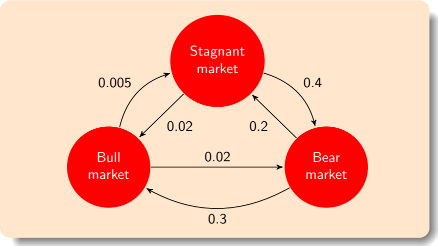
$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

is the **rate matrix**.

"Is it ever more likely to be a Bear market than a Bull market?"

$$\exists t \left(P(t)_{\text{Bear}} \geq P(t)_{\text{Bull}} \right)$$

Reachability for Continuous-Time Markov Chains



Distribution $P(t)$ at time t satisfies $P'(t) = P(t)Q$, where

$$Q = \begin{pmatrix} -0.025 & 0.02 & 0.005 \\ 0.3 & -0.5 & 0.2 \\ 0.02 & 0.4 & -0.42 \end{pmatrix}$$

is the **rate matrix**.

Stationary distribution $\pi = (0.885, 0.071, 0.044)$.

Cyber-Physical Systems

"To analyze a cyber-physical system, such as a pacemaker, we need to consider the **discrete software controller** *interacting with the physical world, which is typically modelled by* **differential equations***"*

Rajeev Alur (CACM, 2013)

Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$

Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow \quad$ timed automata

Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow$ timed automata
  - $\dot{\mathbf{x}} = \mathbf{c} \quad \Rightarrow$ rectangular hybrid automata

Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow \quad$ timed automata
  - $\dot{\mathbf{x}} = \mathbf{c} \quad \Rightarrow \quad$ rectangular hybrid automata
  - $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \Rightarrow \quad$ linear hybrid automata

Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow \quad$ timed automata
  - $\dot{\mathbf{x}} = \mathbf{c} \quad \Rightarrow \quad$ rectangular hybrid automata
  - $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \Rightarrow \quad$ linear hybrid automata
  - $\ldots$

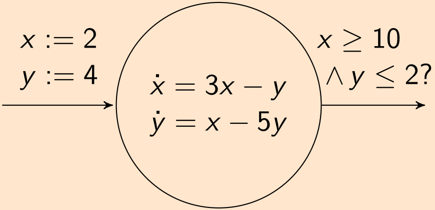
Hybrid Automata: Various Continuous Dynamics

- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow$ timed automata
  - $\dot{\mathbf{x}} = \mathbf{c} \quad \Rightarrow$ rectangular hybrid automata
  - $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \Rightarrow$ linear hybrid automata
  - ...
- o-minimal flows + strong resets $\Rightarrow$ reachability decidable

Hybrid Automata: Various Continuous Dynamics

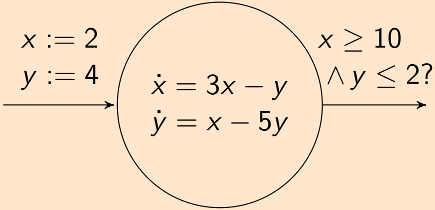
- Hybrid automaton = states + variables $\mathbf{x} \in \mathbb{R}^k$
  - $\dot{\mathbf{x}} = \mathbf{1} \quad \Rightarrow$ timed automata
  - $\dot{\mathbf{x}} = \mathbf{c} \quad \Rightarrow$ rectangular hybrid automata
  - $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \quad \Rightarrow$ linear hybrid automata
  - $\ldots$
- o-minimal flows + strong resets $\Rightarrow$ reachability decidable

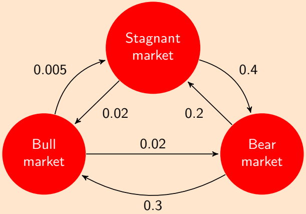
Is this location a trap?



Reachability for Continuous Linear Dynamical Systems

Is this location a trap?





Is ever more likely to be a Bear market than a Bull market:

$$\exists t \left(P(t)_{\mathrm{Bear}} \geq P(t)_{\mathrm{Bull}} \right) ?$$

Reachability for Continuous Linear Dynamical Systems

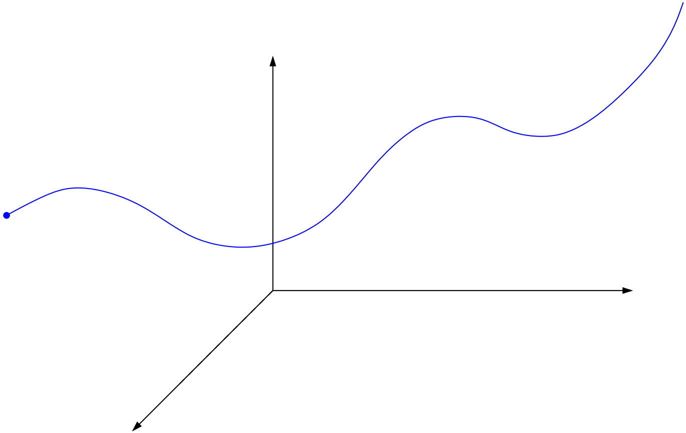
$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

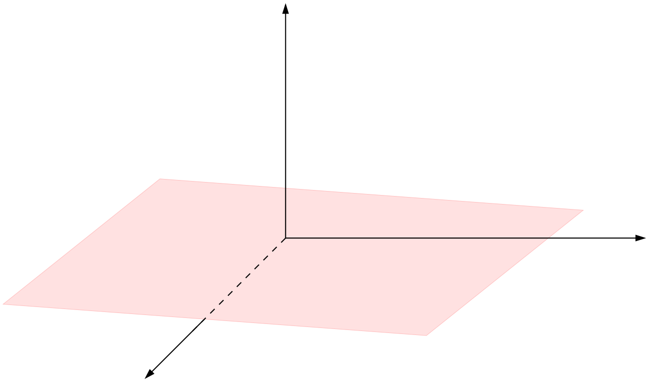


Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

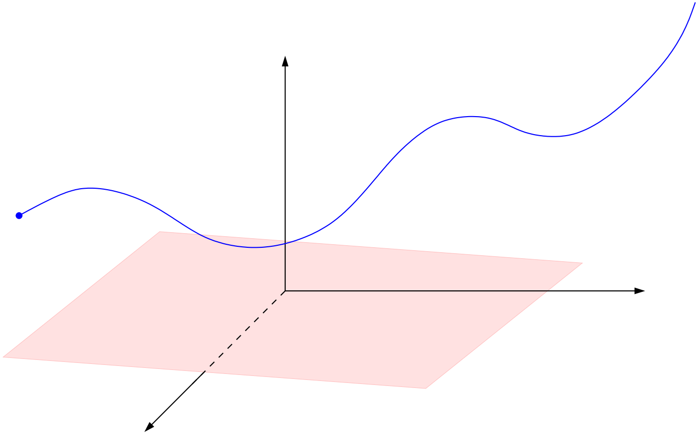


Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$

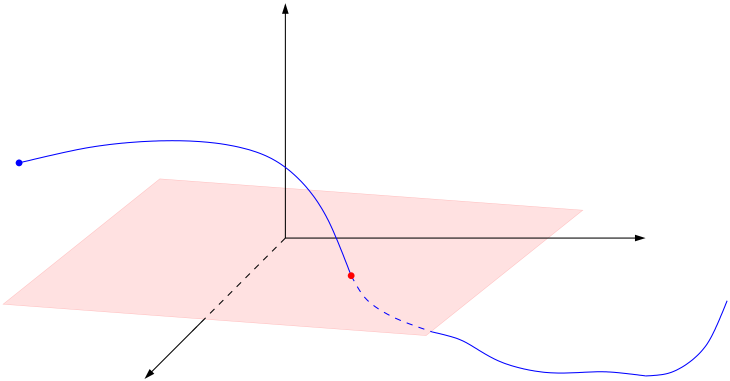
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$



Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

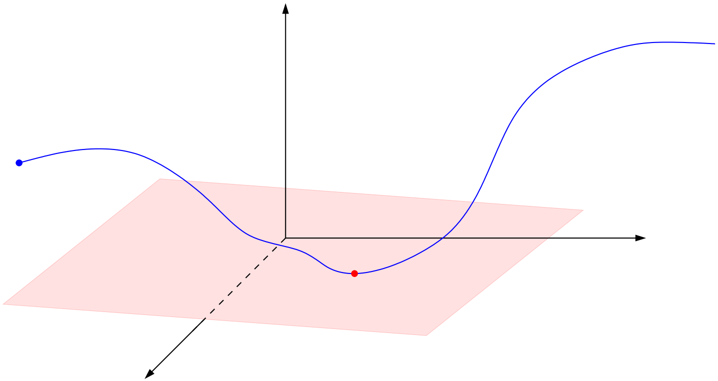


Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$

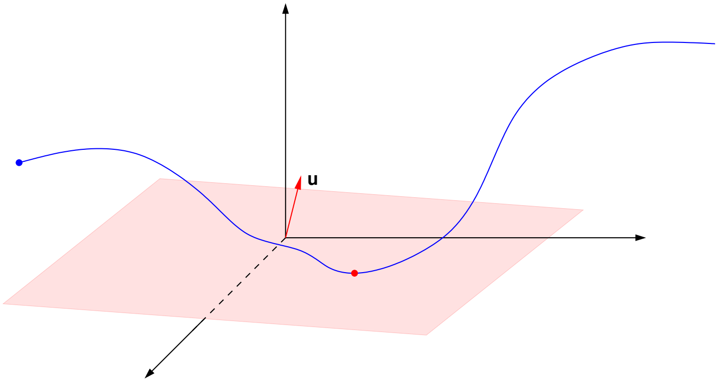
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$



Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$



Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

$$f(t) = \mathbf{u}^{\mathsf{T}}\mathbf{x}(t)$$

Reachability for Continuous Linear Dynamical Systems

$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$

$$f(t) = \mathbf{u}^{\mathsf{T}}\mathbf{x}(t)$$

$$f^{(k)}(t) + a_{k-1}f^{(k-1)}(t) + \ldots + a_1 f'(t) + a_0 f(t) = 0$$

Reachability for Continuous Linear Dynamical Systems

$$\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^k$$
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\Rightarrow \quad \mathbf{x}(t) = \exp(\mathbf{A}t)\mathbf{x}(0)$$

$$f(t) = \mathbf{u}^{\mathsf{T}}\mathbf{x}(t)$$

$$f^{(k)}(t) + a_{k-1}f^{(k-1)}(t) + \ldots + a_1 f'(t) + a_0 f(t) = 0$$

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$$

Reachability for Continuous Linear Dynamical Systems

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

Reachability for Continuous Linear Dynamical Systems

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

BOUNDED-ZERO Problem

Instance: f and bounded interval $[a, b]$
Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

Reachability for Continuous Linear Dynamical Systems

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

BOUNDED-ZERO Problem

Instance: f and bounded interval $[a, b]$
Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

ZERO Problem

Instance: f
Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

Reachability for Continuous Linear Dynamical Systems

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given as above, with all coefficients algebraic.

BOUNDED-ZERO Problem

Instance: f and bounded interval $[a, b]$
Question: Is there $t \in [a, b]$ such that $f(t) = 0$?

ZERO Problem

Instance: f
Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

- **Decidability open!** [Bell, Delvenne, Jungers, Blondel 2010]

Related Work

A lot of work since 1920s on the zeros of exponential polynomials

$$f(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z}$$

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, . . .)
but mostly on distribution of *complex* zeros.

Related Work

A lot of work since 1920s on the zeros of exponential polynomials

$$f(z) = \sum_{j=1}^{m} P_j(z) e^{\lambda_j z}$$

(Polya, Ritt, Tamarkin, Kac, Voorhoeve, van der Poorten, ...) but mostly on distribution of *complex* zeros.

CONTINUOUS-ORBIT Problem

The problem of whether the trajectory $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ reaches a given **target point** was shown to be decidable by Hainry (2008) and in PTIME by Chen, Han and Yu (2015).

Reachability for Continuous Linear Dynamical Systems

Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

Reachability for Continuous Linear Dynamical Systems

Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

Theorem (arXiv:1507.03632, 2015)

In dimension 3, BOUNDED-ZERO and ZERO are decidable.

Reachability for Continuous Linear Dynamical Systems

Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

Theorem (arXiv:1507.03632, 2015)

In dimension 3, BOUNDED-ZERO and ZERO are decidable.

Theorem (arXiv:1506.00695, 2015)

Assuming Schanuel's Conjecture, BOUNDED-ZERO is decidable in all dimensions.

Reachability for Continuous Linear Dynamical Systems

Theorem (Bell, Delvenne, Jungers, Blondel 2010)

In dimension 2, BOUNDED-ZERO and ZERO are decidable.

Theorem (arXiv:1507.03632, 2015)

In dimension 3, BOUNDED-ZERO and ZERO are decidable.

Theorem (arXiv:1506.00695, 2015)

Assuming Schanuel's Conjecture, BOUNDED-ZERO is decidable in all dimensions.

It turns out that this result (in fact, a powerful generalisation of it) had already been discovered (but never published) in the early 1990s by Macintyre and Wilkie!

[Angus Macintyre, personal communication, July 2015]

Reachability for Continuous Linear Dynamical Systems

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

Reachability for Continuous Linear Dynamical Systems

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

Theorem (arXiv:1506.00695, 2015)

In dimension 9 (and above), decidability of ZERO would entail major breakthroughs in Diophantine approximation—the Diophantine approximation type of α would be computable to within arbitrary precision.

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

Schanuel's Conjecture

If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$ then some n-element subset of $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$ is algebraically independent.

Schanuel's Conjecture

Theorem (Lindemann-Weierstrass)

If $a_1, \ldots, a_n$ are algebraic numbers linearly independent over $\mathbb{Q}$, then $e^{a_1}, \ldots, e^{a_n}$ are algebraically independent.

Schanuel's Conjecture

If $z_1, \ldots, z_n$ are complex numbers linearly independent over $\mathbb{Q}$ then some n-element subset of $\{z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}\}$ is algebraically independent.

Example

By Schanuel's conjecture some two-element subset of $\{1, \pi i, e^1, e^{\pi i}\}$ is algebraically independent.

The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$

The BOUNDED-ZERO Problem

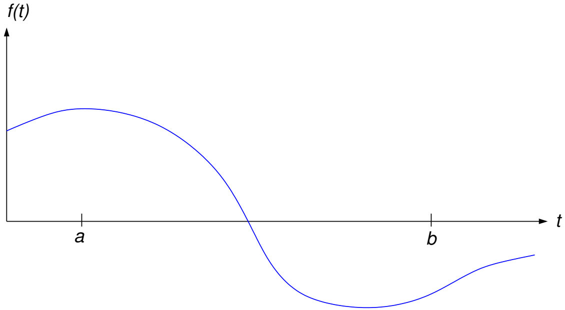
Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$

The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$

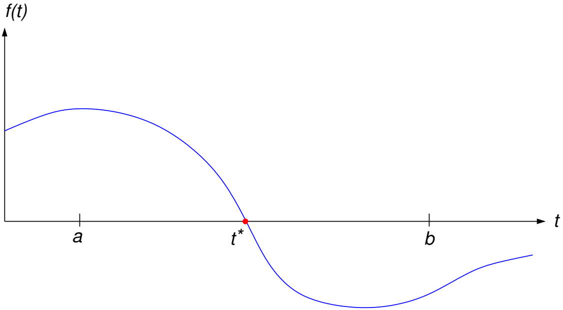
The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$



The BOUNDED-ZERO Problem

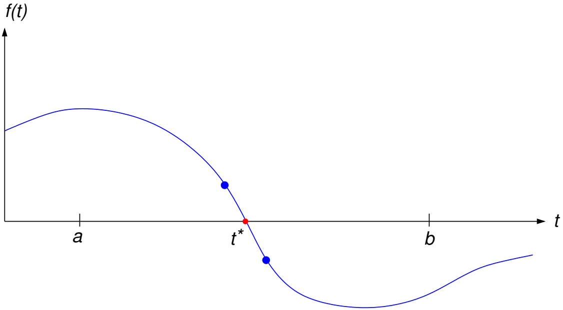
Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$



'non-trivial' zero $\Rightarrow$ t^* transcendental

The BOUNDED-ZERO Problem

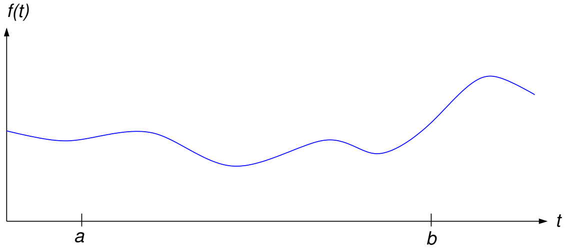
Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$



'non-trivial' zero $\Rightarrow$ t^* transcendental

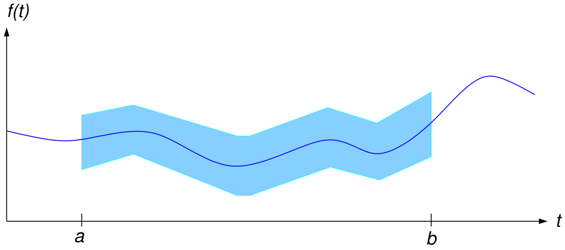
The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$



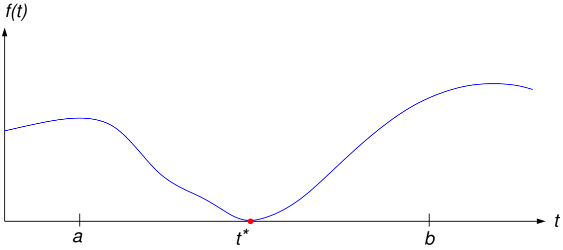
The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$



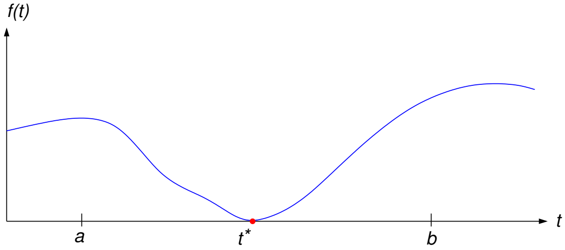
The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$



The BOUNDED-ZERO Problem

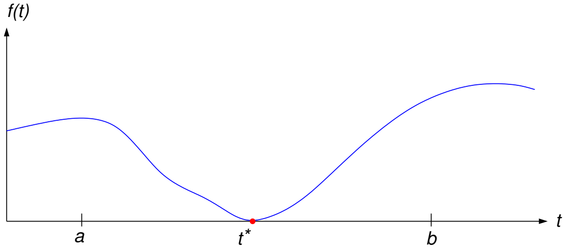
Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t)e^{\lambda_j t}$



Can this situation arise?

The BOUNDED-ZERO Problem

Real-valued exponential polynomial $f(t) = \sum_{j=1}^{m} P_j(t) e^{\lambda_j t}$



Easily! For example, $f(t) = 2 + e^{it} + e^{-it}$.

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1} \,.$$

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1}\,.$$

- Factorisation $P(z) = (1+z)(1+z^{-1})$ induces a factorisation

$$f(t) = \underbrace{(1 + e^{it})}_{f_1(t)}\underbrace{(1 - e^{it})}_{f_2(t)}$$

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1}.$$

- Factorisation $P(z) = (1 + z)(1 + z^{-1})$ induces a factorisation

$$f(t) = \underbrace{(1 + e^{it})}_{f_1(t)}\underbrace{(1 - e^{it})}_{f_2(t)}$$

- Common zeros of f_1 and f_2 are tangential zeros of f

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1}\,.$$

- Factorisation $P(z) = (1+z)(1+z^{-1})$ induces a factorisation

$$f(t) = \underbrace{(1 + e^{it})}_{f_1(t)}\underbrace{(1 - e^{it})}_{f_2(t)}$$

- Common zeros of f_1 and f_2 are tangential zeros of f

Idea: factorise f.

Laurent Polynomials and Factorisation

Example

- Write $f(t) = 2 + e^{it} + e^{-it}$ in the form $f(t) = P(e^{it})$ for the **Laurent polynomial**

$$P(z) = 2 + z + z^{-1}\,.$$

- Factorisation $P(z) = (1 + z)(1 + z^{-1})$ induces a factorisation

$$f(t) = \underbrace{(1 + e^{it})}_{f_1(t)}\underbrace{(1 - e^{it})}_{f_2(t)}$$

- Common zeros of f_1 and f_2 are tangential zeros of f

Idea: factorise f. Noting that factors may be complex-valued!

The Real Case

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$.

The Real Case

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$.

Lemma

Assuming Schanuel's conjecture, if f is real valued and P is irreducible then f has no tangential zeros.

The Real Case

Any exponential polynomial $f(t)$ can be written

$$f(t) = P(t, e^{a_1 t}, \ldots, e^{a_m t})$$

with

$$P \in \mathbb{C}[x, x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

and $\{a_1, \ldots, a_m\}$ a set of real and imaginary algebraic numbers that is linearly independent over $\mathbb{Q}$.

Lemma

Assuming Schanuel's conjecture, if f is real valued and P is irreducible then f has no tangential zeros.

Complex case requires some new ideas . . .

The Unbounded Case

"there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know."

Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

Continued Fractions

Finite continued fractions:

$$
\begin{aligned}
[3, 7, 15, 1, 292] &= 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}} \\
&= 3.141592653\ldots
\end{aligned}
$$

Continued Fractions

Finite continued fractions:

$$[3, 7, 15, 1, 292] = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

$$= 3.141592653\ldots$$

Infinite continued fractions:

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

$$\sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots]$$

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

$$\sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots]$$

What about numbers of degree ≥ 3?

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

$$\sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots]$$

What about numbers of degree ≥ 3?

$$\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1$$
$$3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, \ldots]$$

Real Algebraic Numbers

Theorem

The continued fraction expansion of a real quadratic irrational number is periodic.

$$\sqrt{389} = [19, 1, 2, 1, 1, 1, 1, 2, 1, 38, 1, 2, 1, 1, 1, 1, 2, 1, 38, \ldots]$$

What about numbers of degree ≥ 3?

$$\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, 1\ 3, 4, 1, 1, 2, 14, 3, 12, 1, 15, 3, 1, 4, 534, 1, 1, 5, 1, 1, \ldots]$$

Lang and Trotter: "*no significant departure from random behaviour*"

An Open Problem

" [. . .] no continued fraction development of an algebraic number of higher degree than the second is known. It is not even known if such a development has bounded elements."

A. Khinchin. 1949.

An Open Problem

" [. . .] no continued fraction development of an algebraic number of higher degree than the second is known. It is not even known if such a development has bounded elements."

A. Khinchin. 1949.

"Is there an algebraic number of degree higher than two whose simple continued fraction has unbounded partial quotients? Does every such number have unbounded partial quotients?"

R. K. Guy, 2004

A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \ldots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$.

A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \ldots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$.

Theorem (arXiv:1506.00695, 2015)

If the ZERO PROBLEM is decidable at dimension 9 then

$$\{x \in \mathbb{R} \cap \mathbb{A} : S(x) < \infty\}$$

is recursively enumerable.

A Mathematical Obstacle at Dimension 9

Given $x = [a_0, a_1, a_2, \ldots]$, define $S(x) = \sup_{n \in \mathbb{N}} a_n$.

Theorem (arXiv:1506.00695, 2015)

If the ZERO PROBLEM is decidable at dimension 9 then

$$\{x \in \mathbb{R} \cap \mathbb{A} : S(x) < \infty\}$$

is recursively enumerable.

Remark

Perhaps this set is recursive—it may even be $\emptyset$ or $\mathbb{R} \cap \mathbb{A}$. However proving recursive enumerability would be a significant achievement.

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left| x - \frac{m}{n} \right|$$

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left| x - \frac{m}{n} \right|$$

Theorem (Dirichlet 1842)

There are infinitely many integers m, n such that $\left| x - \frac{m}{n} \right| < \frac{1}{n^2}$.

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left|x - \frac{m}{n}\right|$$

Theorem (Dirichlet 1842)

There are infinitely many integers m, n such that $\left|x - \frac{m}{n}\right| < \frac{1}{n^2}$.

- $S(x) < \infty$ if and only if there exists $\varepsilon > 0$ such that

$$\left|x - \frac{m}{n}\right| < \frac{\varepsilon}{n^2}$$

has no solutions.

Diophantine Approximation

How well can one approximate a real number x with rationals?

$$\left|x - \frac{m}{n}\right|$$

Theorem (Dirichlet 1842)

There are infinitely many integers m, n such that $\left|x - \frac{m}{n}\right| < \frac{1}{n^2}$.

- $S(x) < \infty$ if and only if there exists $\varepsilon > 0$ such that

  $$\left|x - \frac{m}{n}\right| < \frac{\varepsilon}{n^2}$$

  has no solutions.

- Relate this to the existence of zeros of order-9 exponential polynomial $f(t)$ with terms e^{ixt} and e^{it}.

The ZERO Problem

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

The ZERO Problem

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

The ZERO Problem

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

- Diophantine approximation
  - Kronecker's Theorem on simultaneous Diophantine approximation.

The ZERO Problem

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

- Diophantine approximation
  - Kronecker's Theorem on simultaneous Diophantine approximation.
  - Baker's Theorem on lower bounds for **linear forms in logarithms** of algebraic numbers.

The ZERO Problem

ZERO Problem

Instance: f

Question: Is there $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$?

Theorem (arXiv:1507.03632, 2015)

In dimension 8 or less, ZERO reduces to BOUNDED-ZERO.

- Diophantine approximation
  - Kronecker's Theorem on simultaneous Diophantine approximation.
  - Baker's Theorem on lower bounds for **linear forms in logarithms** of algebraic numbers.
- Model theory of the reals
  - **o-minimality** of $(\mathbb{R}, <, +, \times, e^x, 0, 1)$.

Conclusion and Perspectives

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

for all $n \geq 0$.

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

for all $n \geq 0$.

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

The Discrete Case

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of integers such that there exist constants $a_1, \ldots, a_k$, such that

$$u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n$$

for all $n \geq 0$.

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully constructive.

The Skolem Problem

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

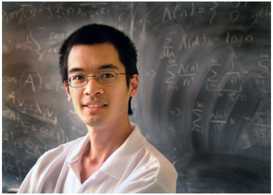
The Skolem Problem

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

"It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for 'linear' automata!"

Terence Tao



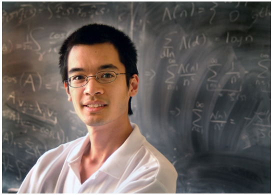
The Skolem Problem

Skolem Problem

Does $\exists n$ such that $u_n = 0$?

"It is faintly outrageous that this problem is still open; it is saying that we do not know how to decide the Halting Problem even for 'linear' automata!"

Terence Tao



"... a mathematical embarrassment ..."

Richard Lipton

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Not a mathematical embarrassment!

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Not a mathematical embarrassment!
- Even the bounded problem is hard (apparently).

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Not a mathematical embarrassment!
- Even the bounded problem is hard (apparently).
- Formidable "mathematical obstacle" at dimension 9 in the unbounded case.

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Not a mathematical embarrassment!
- Even the bounded problem is hard (apparently).
- Formidable "mathematical obstacle" at dimension 9 in the unbounded case.
- The infinite-zeros problem is also hard.

Wrapping Things Up

Continuous Skolem Problem

Does $\exists t$ such that $f(t) = 0$?

- Not a mathematical embarrassment!
- Even the bounded problem is hard (apparently).
- Formidable "mathematical obstacle" at dimension 9 in the unbounded case.
- The infinite-zeros problem is also hard.
- Diophantine-approximation techniques unavoidable.